

# Borell's generalized Prékopa-Leindler inequality: A simple proof

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## Abstract

We present a simple proof of Christer Borell's general inequality in the Brunn-Minkowski theory. We then discuss applications of Borell's inequality to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

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## 1 Introduction

Let us denote by  $\text{supp}(f)$  the support of a function  $f$ . In [6] Christer Borell proved the following inequality (see [6, Theorem 2.1]), which we will call the Borell-Brunn-Minkowski inequality.

**Theorem 1** (Borell-Brunn-Minkowski inequality). *Let  $f, g, h : \mathbb{R}^n \rightarrow [0, +\infty)$  be measurable functions. Let  $\varphi = (\varphi_1, \dots, \varphi_n) : \text{supp}(f) \times \text{supp}(g) \rightarrow \mathbb{R}^n$  be a continuously differentiable function with positive partial derivatives, such that  $\varphi_k(x, y) = \varphi_k(x_k, y_k)$  for every  $x = (x_1, \dots, x_n) \in \text{supp}(f)$ ,  $y = (y_1, \dots, y_n) \in \text{supp}(g)$ . Let  $\Phi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$  be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$h(\varphi(x, y)) \prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x) \prod_{k=1}^n \rho_k, g(y) \prod_{k=1}^n \eta_k) \quad (1)$$

*holds for every  $x \in \text{supp}(f)$ , for every  $y \in \text{supp}(g)$ , for every  $\rho_1, \dots, \rho_n > 0$  and for every  $\eta_1, \dots, \eta_n > 0$ , then*

$$\int h \geq \Phi \left( \int f, \int g \right).$$

C. Borell proved a slightly more general statement, involving an arbitrary number of functions. For simplicity, we restrict ourselves to the statement of Theorem 1.

Theorem 1 yields several important consequences. For example, applying Theorem 1 to indicators of compact sets (i.e.  $f = 1_A$ ,  $g = 1_B$ ,  $h = 1_{\varphi(A, B)}$ ) yields the following generalized Brunn-Minkowski inequality.

**Corollary 2** (Generalized Brunn-Minkowski inequality). *Let  $A, B$  be compact subsets of  $\mathbb{R}^n$ . Let  $\varphi = (\varphi_1, \dots, \varphi_n) : A \times B \rightarrow \mathbb{R}^n$  be a continuously differentiable function with positive partial derivatives, such that  $\varphi_k(x, y) = \varphi_k(x_k, y_k)$  for every  $x = (x_1, \dots, x_n) \in A$ ,  $y = (y_1, \dots, y_n) \in B$ . Let  $\Phi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$  be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$\prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\prod_{k=1}^n \rho_k, \prod_{k=1}^n \eta_k)$$

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holds for every  $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$ , then

$$|\varphi(A, B)| \geq \Phi(|A|, |B|),$$

where  $|\cdot|$  denotes Lebesgue measure and  $\varphi(A, B) = \{\varphi(x, y) : x \in A, y \in B\}$ .

The classical Brunn-Minkowski inequality (see e.g. [23], [13]) follows from Corollary 2 by taking  $\varphi(x, y) = x + y$ ,  $x \in A, y \in B$ , and  $\Phi(a, b) = (a^{1/n} + b^{1/n})^n$ ,  $a, b \geq 0$ . Although the Brunn-Minkowski inequality goes back to more than a century ago, it still attracts a lot of attention (see e.g. [20], [11], [14], [18], [9], [10], [12], [15], [17]).

Theorem 1 also allows us to recover the so-called Borell-Brascamp-Lieb inequality. Let us denote by  $M_s^\lambda(a, b)$  the  $s$ -mean of the real numbers  $a, b \geq 0$  with weight  $\lambda \in [0, 1]$ , defined as

$$M_s^\lambda(a, b) = ((1 - \lambda)a^s + \lambda b^s)^{\frac{1}{s}} \quad \text{if } s \notin \{-\infty, 0, +\infty\},$$

$M_{-\infty}^\lambda(a, b) = \min(a, b)$ ,  $M_0^\lambda(a, b) = a^{1-\lambda}b^\lambda$ ,  $M_{+\infty}^\lambda(a, b) = \max(a, b)$ . We will need the following Hölder inequality (see e.g. [16]).

**Lemma 3** (Generalized Hölder inequality). *Let  $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$  such that  $\beta + \gamma \geq 0$  and  $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$ . Then, for every  $a, b, c, d \geq 0$  and  $\lambda \in [0, 1]$ ,*

$$M_\alpha^\lambda(ac, bd) \leq M_\beta^\lambda(a, b)M_\gamma^\lambda(c, d).$$

**Corollary 4** (Borell-Brascamp-Lieb inequality). *Let  $\gamma \geq -\frac{1}{n}$ ,  $\lambda \in [0, 1]$  and  $f, g, h : \mathbb{R}^n \rightarrow [0, +\infty)$  be measurable functions. If the inequality*

$$h((1 - \lambda)x + \lambda y) \geq M_\gamma^\lambda(f(x), g(y))$$

*holds for every  $x \in \text{supp}(f), y \in \text{supp}(g)$ , then*

$$\int_{\mathbb{R}^n} h \geq M_{\frac{\gamma}{1+\gamma n}}^\lambda \left( \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).$$

Corollary 4 follows from Theorem 1 by taking  $\varphi(x, y) = (1 - \lambda)x + \lambda y$ ,  $x \in \text{supp}(f), y \in \text{supp}(g)$ , and  $\Phi(a, b) = M_{\frac{\gamma}{1+\gamma n}}^\lambda(a, b)$ ,  $a, b \geq 0$ . Indeed, using Lemma 3, one obtains that for every  $x \in \text{supp}(f), y \in \text{supp}(g)$ , and for every  $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$ ,

$$\begin{aligned} h(\varphi(x, y)) \Pi_{k=1}^n \left( \frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) &= h((1 - \lambda)x + \lambda y) \Pi_{k=1}^n ((1 - \lambda)\rho_k + \lambda \eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) M_{\frac{1}{n}}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k) \\ &\geq M_{\frac{\gamma}{1+\gamma n}}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k) \\ &= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k). \end{aligned}$$

Corollary 4 was independently proved by Borell (see [6, Theorem 3.1]), and by Brascamp and Lieb [8].

Another important consequence of the Borell-Brunn-Minkowski inequality is obtained when considering  $\varphi$  to be nonlinear. Let us denote for  $\mathbf{p} = (p_1, \dots, p_n) \in [-\infty, +\infty]^n$ ,  $x = (x_1, \dots, x_n) \in [0, +\infty]^n$  and  $y = (y_1, \dots, y_n) \in [0, +\infty]^n$ ,

$$M_{\mathbf{p}}^\lambda(x, y) = (M_{p_1}^\lambda(x_1, y_1), \dots, M_{p_n}^\lambda(x_n, y_n)).$$

**Corollary 5** (nonlinear extension of the Brunn-Minkowski inequality). *Let  $\mathbf{p} = (p_1, \dots, p_n) \in [0, 1]^n$ ,  $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$ ,  $\lambda \in [0, 1]$ , and  $f, g, h : [0, +\infty)^n \rightarrow [0, +\infty)$  be measurable functions. If the inequality*

$$h(M_{\mathbf{p}}^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y))$$

*holds for every  $x \in \text{supp}(f), y \in \text{supp}(g)$ , then*

$$\int_{[0, +\infty)^n} h \geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda \left( \int_{[0, +\infty)^n} f, \int_{[0, +\infty)^n} g \right).$$

Corollary 5 follows from Theorem 1 by taking  $\varphi(x, y) = M_{\mathbf{p}}^\lambda(x, y)$ ,  $x \in \text{supp}(f)$ ,  $y \in \text{supp}(g)$ , and  $\Phi(a, b) = M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda(a, b)$ ,  $a, b \geq 0$ . Indeed, using Lemma 3, one obtains that for every  $x \in \text{supp}(f)$ ,  $y \in \text{supp}(g)$ , and for every  $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$ ,

$$\begin{aligned} h(\varphi(x, y)) \Pi_{k=1}^n \left( \frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) &= h(M_{\mathbf{p}}^\lambda(x, y)) \Pi_{k=1}^n M_{\frac{p_k}{1-p_k}}^\lambda(x_k^{1-p_k}, y_k^{1-p_k}) M_1(x_k^{p_k-1} \rho_k, y_k^{p_k-1} \eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) \Pi_{k=1}^n M_{p_k}^\lambda(\rho_k, \eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) M_{(\sum_{i=1}^n p_i^{-1})^{-1}}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k) \\ &\geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k) \\ &= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k). \end{aligned}$$

In the particular case where  $\mathbf{p} = (0, \dots, 0)$ , Corollary 5 was rediscovered by Ball [1]. In the general case, Corollary 5 was rediscovered by Uhrin [24].

Notice that the condition on  $p$  in Corollary 5 is less restrictive in dimension 1. It reads as follows:

**Corollary 6** (nonlinear extension of the Brunn-Minkowski inequality on the line). *Let  $p \leq 1$ ,  $\gamma \geq -p$ , and  $\lambda \in [0, 1]$ . Let  $f, g, h : [0, +\infty) \rightarrow [0, +\infty)$  be measurable functions such that for every  $x \in \text{supp}(f)$ ,  $y \in \text{supp}(g)$ ,*

$$h(M_p^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y)).$$

Then,

$$\int_0^{+\infty} h \geq M_{(\frac{1}{p} + \frac{1}{\gamma})^{-1}}^\lambda \left( \int_0^{+\infty} f, \int_0^{+\infty} g \right).$$

A simple proof of Corollary 6 was recently given by Bobkov et al. [4].

In section 2, we present a simple proof of Theorem 1, based on mass transportation. In section 3, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang. We also prove an equivalence between the log-Brunn-Minkowski inequality and its possible extensions to convex measures (see section 3 for precise definitions).

## 2 A simple proof of the Borell-Brunn-Minkowski inequality

In this section, we present a simple proof of Theorem 1.

*Proof of Theorem 1.* The proof is done by induction on the dimension. To prove the theorem in dimension 1, we use a mass transportation argument.

**Step 1 :** (In dimension 1)

First let us see that if  $\int f = 0$  or  $\int g = 0$ , then the result holds. Let us assume, without loss of generality, that  $\int g = 0$ . By taking  $\rho = 1$ , by letting  $\eta$  go to 0 and by using continuity and homogeneity of  $\Phi$  in the condition (1), one obtains

$$h(\varphi(x, y)) \frac{\partial \varphi}{\partial x} \geq \Phi(f(x), 0) = f(x) \Phi(1, 0).$$

It follows that, for fixed  $y \in \text{supp}(g)$ ,

$$\int h(z) dz \geq \int_{\varphi(\text{supp}(f), y)} h(z) dz = \int_{\text{supp}(f)} h(\varphi(x, y)) \frac{\partial \varphi}{\partial x} dx \geq \int f \Phi(1, 0) = \Phi \left( \int f, \int g \right).$$

A similar argument shows that the result holds if  $\int f = +\infty$  or  $\int g = +\infty$ . Thus we assume thereafter that  $0 < \int f < +\infty$  and  $0 < \int g < +\infty$ .

Let us show that one may assume that  $\int f = \int g = 1$ . Let us define, for  $x, y \in \mathbb{R}$  and  $a, b \geq 0$ ,

$$\begin{aligned}\tilde{f}(x) &= f\left(\Phi\left(\int f, 0\right)x\right)\Phi(1, 0), & \tilde{g}(x) &= g\left(\Phi\left(0, \int g\right)x\right)\Phi(0, 1), \\ \tilde{h}(x) &= h\left(\Phi\left(\int f, \int g\right)x\right), \\ \tilde{\varphi}(x, y) &= \frac{\varphi(\Phi(\int f, 0)x, \Phi(0, \int g)y)}{\Phi(\int f, \int g)}, & \tilde{\Phi}(a, b) &= \Phi\left(a\frac{\int f}{\Phi(\int f, \int g)}, b\frac{\int g}{\Phi(\int f, \int g)}\right).\end{aligned}$$

Let  $x \in \text{supp}(\tilde{f})$ ,  $y \in \text{supp}(\tilde{g})$ , and let  $\tilde{\rho}, \tilde{\eta} > 0$ . One has,

$$\begin{aligned}\tilde{h}(\tilde{\varphi}(x, y))\left(\frac{\partial \tilde{\varphi}}{\partial x}\tilde{\rho} + \frac{\partial \tilde{\varphi}}{\partial y}\tilde{\eta}\right) &\geq \Phi\left(f(\Phi(\int f, 0)x)\frac{\Phi(\int f, 0)}{\Phi(\int f, \int g)}\tilde{\rho}, g(\Phi(0, \int g)y)\frac{\Phi(0, \int g)}{\Phi(\int f, \int g)}\tilde{\eta}\right) \\ &= \tilde{\Phi}(\tilde{f}(x)\tilde{\rho}, \tilde{g}(y)\tilde{\eta}).\end{aligned}$$

Notice that the functions  $\tilde{\varphi}$  and  $\tilde{\Phi}$  satisfy the same assumptions as the functions  $\varphi$  and  $\Phi$  respectively, and that  $\int \tilde{f} = \int \tilde{g} = 1$ . If the result holds for functions of integral one, then

$$\int \tilde{h}(w)dw \geq \tilde{\Phi}(1, 1) = 1.$$

The change of variable  $w = z/\Phi(\int f, \int g)$  leads us to

$$\int h(z)dz \geq \Phi\left(\int f, \int g\right).$$

Assume now that  $\int f = \int g = 1$ . By standard approximation, one may assume that  $f$  and  $g$  are compactly supported positive Lipschitz functions (relying on the fact that  $\Phi$  is continuous and increasing in each coordinate, compare with [2, page 343]). Thus there exists a non-decreasing map  $T : \text{supp}(f) \rightarrow \text{supp}(g)$  such that for every  $x \in \text{supp}(f)$ ,

$$f(x) = g(T(x))T'(x),$$

see e.g. [3], [25]. Since  $T$  is non-decreasing and  $\partial\varphi/\partial x, \partial\varphi/\partial y > 0$ , the function  $\Theta : \text{supp}(f) \rightarrow \varphi(\text{supp}(f), T(\text{supp}(f)))$  defined by  $\Theta(x) = \varphi(x, T(x))$  is bijective. Hence the change of variable  $z = \Theta(x)$  is admissible and one has,

$$\begin{aligned}\int h(z)dz &\geq \int_{\text{supp}(f)} h(\varphi(x, T(x)))\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}T'(x)\right)dx \geq \int_{\text{supp}(f)} \Phi(f(x), g(T(x))T'(x))dx \\ &= \int \Phi(f(x), f(x))dx.\end{aligned}$$

Using homogeneity of  $\Phi$ , one deduces that

$$\int h \geq \Phi(1, 1) \int f(x)dx = \Phi\left(\int f, \int g\right).$$

## Step 2 : (Tensorization)

Let  $n$  be a positive integer and assume that Theorem 1 holds in  $\mathbb{R}^n$ . Let  $f, g, h, \varphi, \Phi$  satisfying the assumptions of Theorem 1 in  $\mathbb{R}^{n+1}$ . Recall that the inequality

$$h(\varphi(x, y))\prod_{k=1}^{n+1}\left(\frac{\partial \varphi_k}{\partial x_k}\rho_k + \frac{\partial \varphi_k}{\partial y_k}\eta_k\right) \geq \Phi(f(x)\prod_{k=1}^{n+1}\rho_k, g(y)\prod_{k=1}^{n+1}\eta_k), \quad (2)$$

holds for every  $x \in \text{supp}(f), y \in \text{supp}(g)$ , and for every  $\rho_1, \dots, \rho_{n+1}, \eta_1, \dots, \eta_{n+1} > 0$ . Let us define, for  $x_{n+1}, y_{n+1}, z_{n+1} \in \mathbb{R}$ ,

$$F(x_{n+1}) = \int_{\mathbb{R}^n} f(x, x_{n+1}) dx, \quad G(y_{n+1}) = \int_{\mathbb{R}^n} g(y, y_{n+1}) dy, \quad H(z_{n+1}) = \int_{\mathbb{R}^n} h(x, z_{n+1}) dx.$$

Since  $\int f > 0, \int g > 0$ , the support of  $F$  and the support of  $G$  are nonempty. Let  $x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G)$ , and let  $\rho_{n+1}, \eta_{n+1} > 0$ . Let us define, for  $x, y, z \in \mathbb{R}^n$ ,

$$f_{x_{n+1}}(x) = f(x, x_{n+1})\rho_{n+1}, \quad g_{y_{n+1}}(y) = g(y, y_{n+1})\eta_{n+1}, \quad \overline{\varphi}(x, y) = (\varphi_1(x_1, y_1), \dots, \varphi_n(x_n, y_n)),$$

$$h_{\varphi_{n+1}}(z) = h(z, \varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right).$$

Let  $x \in \text{supp}(f_{x_{n+1}}), y \in \text{supp}(g_{y_{n+1}})$ , and let  $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$ . One has

$$\begin{aligned} h_{\varphi_{n+1}}(\overline{\varphi}(x, y)) \Pi_{k=1}^n \left( \frac{\partial \overline{\varphi}_k}{\partial x_k} \rho_k + \frac{\partial \overline{\varphi}_k}{\partial y_k} \eta_k \right) &= h(\varphi(x, x_{n+1}, y, y_{n+1})) \Pi_{k=1}^{n+1} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \\ &\geq \Phi(f(x, x_{n+1}) \Pi_{k=1}^{n+1} \rho_k, g(y, y_{n+1}) \Pi_{k=1}^{n+1} \eta_k) \\ &= \Phi(f_{x_{n+1}}(x) \Pi_{k=1}^n \rho_k, g_{y_{n+1}}(y) \Pi_{k=1}^n \eta_k), \end{aligned}$$

where the inequality follows from inequality (2). Hence, applying Theorem 1 in dimension  $n$ , one has

$$\int_{\mathbb{R}^n} h_{\varphi_{n+1}}(x) dx \geq \Phi \left( \int_{\mathbb{R}^n} f_{x_{n+1}}(x) dx, \int_{\mathbb{R}^n} g_{y_{n+1}}(x) dx \right).$$

This yields that for every  $x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G)$ , and for every  $\rho_{n+1}, \eta_{n+1} > 0$ ,

$$H(\varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right) \geq \Phi(F(x_{n+1}), G(y_{n+1})).$$

Hence, applying Theorem 1 in dimension 1, one has

$$\int_{\mathbb{R}} H(x) dx \geq \Phi \left( \int_{\mathbb{R}} F(x) dx, \int_{\mathbb{R}} G(x) dx \right).$$

This yields the desired inequality.  $\square$

### 3 Applications to the log-Brunn-Minkowski inequality

In this section, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang [7].

Recall that a *convex body* in  $\mathbb{R}^n$  is a compact convex subset of  $\mathbb{R}^n$  with nonempty interior. Böröczky et al. conjectured the following inequality.

**Conjecture 7** (log-Brunn-Minkowski inequality). *Let  $K, L$  be symmetric convex bodies in  $\mathbb{R}^n$  and let  $\lambda \in [0, 1]$ . Then,*

$$|(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L| \geq |K|^{1-\lambda} |L|^\lambda.$$

Here,

$$(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_L(u)^\lambda, \text{ for all } u \in S^{n-1}\},$$

where  $S^{n-1}$  denotes the  $n$ -dimensional Euclidean unit sphere,  $h_K$  denotes the support function of  $K$ , defined by  $h_K(u) = \max_{x \in K} \langle x, u \rangle$ , and  $|\cdot|$  stands for Lebesgue measure.

Böröczky et al. [7] proved that Conjecture 7 holds in the plane. Using Corollary 5 with  $\mathbf{p} = (0, \dots, 0)$ , Saroglou [21] proved that Conjecture 7 holds for unconditional convex bodies

in  $\mathbb{R}^n$  (a set  $K \subset \mathbb{R}^n$  is *unconditional* if for every  $(x_1, \dots, x_n) \in K$  and for every  $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$ , one has  $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$ ).

Recall that a measure  $\mu$  is *s-concave*,  $s \in [-\infty, +\infty]$ , if the inequality

$$\mu((1-\lambda)A + \lambda B) \geq M_s^\lambda(\mu(A), \mu(B))$$

holds for all compact sets  $A, B \subset \mathbb{R}^n$  such that  $\mu(A)\mu(B) > 0$  and for every  $\lambda \in [0, 1]$  (see [5], [6]). The 0-concave measures are also called *log-concave measures*, and the  $-\infty$ -concave measures are also called *convex measures*. A function  $f : \mathbb{R}^n \rightarrow [0, +\infty)$  is  $\alpha$ -concave,  $\alpha \in [-\infty, +\infty]$ , if the inequality

$$f((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))$$

holds for every  $x, y \in \mathbb{R}^n$  such that  $f(x)f(y) > 0$  and for every  $\lambda \in [0, 1]$ .

Saroglou [22] recently proved that if the log-Brunn-Minkowski inequality holds, then the inequality

$$\mu((1-\lambda) \cdot K \oplus_0 \lambda \cdot L) \geq \mu(K)^{1-\lambda} \mu(L)^\lambda$$

holds for every symmetric log-concave measure  $\mu$ , for all symmetric convex bodies  $K, L$  in  $\mathbb{R}^n$  and for every  $\lambda \in [0, 1]$ .

An extension of the log-Brunn-Minkowski inequality for convex measures was proposed by the author in [19], and reads as follows:

**Conjecture 8.** *Let  $p \in [0, 1]$ . Let  $\mu$  be a symmetric measure in  $\mathbb{R}^n$  that has an  $\alpha$ -concave density function, with  $\alpha \geq -\frac{p}{n}$ . Then for every symmetric convex body  $K, L$  in  $\mathbb{R}^n$  and for every  $\lambda \in [0, 1]$ ,*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{(\frac{n}{p} + \frac{1}{\alpha})}^\lambda(\mu(K), \mu(L)). \quad (3)$$

Here,

$$(1-\lambda) \cdot K \oplus_p \lambda \cdot L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq M_p^\lambda(h_K(u), h_L(u)), \text{ for all } u \in S^{n-1}\}.$$

In Conjecture 8, if  $\alpha$  or  $p$  is equal to 0, then  $(n/p + 1/\alpha)^{-1}$  is defined by continuity and is equal to 0. Notice that Conjecture 7 is a particular case of Conjecture 8 when taking  $\mu$  to be Lebesgue measure and  $p = 0$ .

By using Corollary 6, we will prove that Conjecture 7 implies Conjecture 8, when  $\alpha \leq 1$ , generalizing Saroglou's result discussed earlier.

**Theorem 9.** *If the log-Brunn-Minkowski inequality holds, then the inequality*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{(\frac{n}{p} + \frac{1}{\alpha})}^\lambda(\mu(K), \mu(L))$$

*holds for every  $p \in [0, 1]$ , for every symmetric measure  $\mu$  in  $\mathbb{R}^n$  that has an  $\alpha$ -concave density function, with  $1 \geq \alpha \geq -\frac{p}{n}$ , for every symmetric convex body  $K, L$  in  $\mathbb{R}^n$  and for every  $\lambda \in [0, 1]$ .*

*Proof.* Let  $K_0, K_1$  be symmetric convex bodies in  $\mathbb{R}^n$  and let  $\lambda \in (0, 1)$ . Let us denote  $K_\lambda = (1-\lambda) \cdot K_0 \oplus_p \lambda \cdot K_1$  and let us denote by  $\psi$  the density function of  $\mu$ . Let us define, for  $t > 0$ ,  $h(t) = |K_\lambda \cap \{\psi \geq t\}|$ ,  $f(t) = |K_0 \cap \{\psi \geq t\}|$  and  $g(t) = |K_1 \cap \{\psi \geq t\}|$ . Notice that

$$\mu(K_\lambda) = \int_{K_\lambda} \psi(x) dx = \int_{K_\lambda} \int_0^{\psi(x)} dt dx = \int_0^{+\infty} |K_\lambda \cap \{\psi \geq t\}| dt = \int_0^{+\infty} h(t) dt.$$

Similarly, one has

$$\mu(K_0) = \int_0^{+\infty} f(t) dt, \quad \mu(K_1) = \int_0^{+\infty} g(t) dt.$$

Let  $t, s > 0$  such that the sets  $\{\psi \geq t\}$  and  $\{\psi \geq s\}$  are nonempty. Let us denote  $L_0 = \{\psi \geq t\}$ ,  $L_1 = \{\psi \geq s\}$  and  $L_\lambda = \{\psi \geq M_\alpha^\lambda(t, s)\}$ . If  $x \in L_0$  and  $y \in L_1$ , then  $\psi((1 - \lambda)x + \lambda y) \geq M_\alpha^\lambda(\psi(x), \psi(y)) \geq M_\alpha^\lambda(t, s)$ . Hence,

$$L_\lambda \supset (1 - \lambda)L_0 + \lambda L_1 \supset (1 - \lambda) \cdot L_0 \oplus_p \lambda \cdot L_1,$$

the last inclusion following from the fact that  $p \leq 1$ . We deduce that

$$K_\lambda \cap L_\lambda \supset ((1 - \lambda) \cdot K_0 \oplus_p \lambda \cdot K_1) \cap ((1 - \lambda) \cdot L_0 \oplus_p \lambda \cdot L_1) \supset (1 - \lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1).$$

Hence,

$$h(M_\alpha^\lambda(t, s)) = |K_\lambda \cap L_\lambda| \geq |(1 - \lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)| \geq M_{\frac{p}{n}}^\lambda(f(t), g(s)),$$

the last inequality is valid for  $p \geq 0$  and follows from the log-Brunn-Minkowski inequality by using homogeneity of Lebesgue measure (see [7, beginning of section 3]). Thus we may apply Corollary 6 to conclude that

$$\mu(K_\lambda) = \int_0^{+\infty} h \geq M_{(\frac{n}{p} + \frac{1}{\alpha})}^\lambda \left( \int_0^{+\infty} f, \int_0^{+\infty} g \right) = M_{(\frac{n}{p} + \frac{1}{\alpha})}^\lambda (\mu(K_0), \mu(K_1)).$$

□

Since the log-Brunn-Minkowski inequality holds true in the plane, we deduce that Conjecture 8 holds true in the plane (with the restriction  $\alpha \leq 1$ ). Notice that Conjecture 8 holds true in the unconditional case as a consequence of Corollary 5 (see [19]).

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